DEFINITIONS

For all these definitions, let G be a graph and v and w vertices in G

Walks

- A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G which starts at v and finishes at w.

  A walk has the form $v_0e_1v_1e_2\ldots v_{n-1}e_nv_n$, where the v_i's are vertices and e_i's are edges and $v_0=v$ and $v_n=w$, and for all i in $\{1, 2, \ldots, n\}$ v_{i-1} and v_i are the endpoints of e_i.

Trails and Paths

- A trail from v to w is a walk from v to w that does not contain a repeated edge.

  Therefore, a walk $v_0e_1v_1e_2\ldots v_{n-1}e_nv_n$ is a trail iff all the e_i's are distinct.

- A (simple) path from v to w is a trail from v to w that does not contain a repeated vertex.

  Therefore, a trail $v_0e_1v_1e_2\ldots v_{n-1}e_nv_n$ is a simple path iff all the v_i's are distinct.

- The meaning of the term "path" has evolved and is somewhat ambiguous. You may find different definitions in different references.

Closed Walks and Circuits

- A closed walk is a walk that starts and ends at the same vertex.
- A circuit is a closed walk with at least one edge and with no repeated edges, i.e. a non-empty trail which must start and end at the same vertex.
- A simple circuit is a circuit whose only repeated vertices are the first and last.

Trivial walks and circuits

Let G be a graph and v and w vertices in G

- The trivial walk from v to v consists of the single vertex v
- A trivial circuit is a trivial walk, i.e. a circuit consisting of a single vertex and no edge.
- A non-trivial circuit is a circuit with at least one edge.

Summary	**Repeated Edge?**	**Repeated Vertex?**	**Starts and Ends at same point?**	**Must contain at least 1 edge?**
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
(Simple) Path	no	no	no	no
Closed Walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple Circuit	no	first and last only	yes	yes

CONNECTEDNESS

For all these definitions, let *G* be a graph and *v* and *w* vertices in *G*

Definitions

- Vertices *v* and *w* are connected iff there is a walk from *v* to *w*.
- The graph *G* is connected iff any two vertices in *G* are connected.
- A graph that is not connected is said to be disconnected.

Properties

- If *G* is connected, then any two distinct vertices of *G* can be connected by a simple path.
- If vertices *v* and *w* are part of a circuit in *G* and one edge is removed from the circuit, then v and w are still connected.
- If *G* is connected and contains a non-trivial circuit, then an edge of the circuit can be removed without disconnecting *G*.

Connected Components

- A graph *H* is a connected component of a graph *G* iff
  - *H* is a subgraph of *G*
  - *H* is connected
  - None of *H*'s vertices are connected to any vertex of G which is not in *H*.
- Any graph is a union of its connected components.

EULER CIRCUITS AND PATHS

Definition

- Let *G* be a graph and *v* and *w* two distinct vertices in *G*.
- An Euler circuit for *G* is a circuit that contains every vertex and every edge of *G*. Each edge is traversed exactly once.
- I.e. an Euler circuit is a sequence of adjacent vertices and edges in *G* that starts and ends at the same vertex, uses every vertex of *G* at least once, and uses every edge of *G* exactly once.
- An Euler path from *v* to *w* is a sequence of adjacent vertices and edges in *G* that starts at *v* and ends at *w*, passes through every vertex of *G* at least once, and traverses every edge of *G* exactly once.

Theorems

- A graph *G* has an Euler circuit iff *G* is connected and every vertex of *G* has even degree.
- There is an Euler path between two distinct vertices *v* and *w* of a graph *G* iff *G* is connected and *v* and *w* have odd degree and all other vertices of *G* have even degree.

HAMILTONIAN CIRCUITS AND PATHS

Definition

- A Hamiltonian circuit for a graph *G* is a simple circuit that contains every vertex of *G*.
- I.e. a Hamiltonian circuit is a sequences of adjacent vertices and distinct edges in which every vertex appears exactly once except for the first and last which are the same.

Theorem

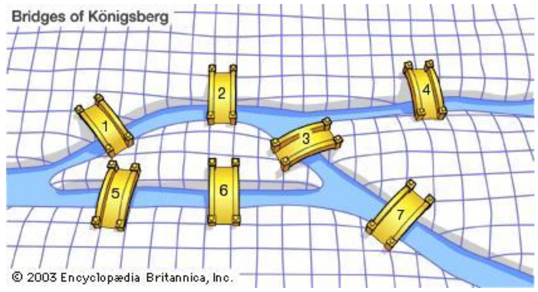
If a graph *G* has a non-trivial Hamiltonian circuit *H*, then *G* has a subgraph *H* with the following properties:

- *H* contains every vertex of *G*.
- *H* is connected
- *H* has the same number of edges as vertices
- Every vertex of *H* has degree 2

PROBLEM OF BRIDGES OF KONIGSBURG

1736: Leonhard Euler gave a paper talking about the following problem:

The town of Konigsberg in Prussia (Kalinigrad in Russia) was built at a point where 2 branches of the Pregel river came together.
It consists of an island and land along the banks. They are connected by 7 bridges.



Q: Is it possible for a person to take a walk around town, starting and ending at the same location and crossing each of the seven bridges exactly once?

ALGORITHM TO BUILD AN EULER CIRCUIT

In both functions G consists of V(G) and E(G)

```
BuildEuler(graph G) // returns an Euler circuit for G {
    pick a vertex v at random from V(G)
    return EulerCircuit(G,v)
    }

EulerCircuit(graph G, vertex v)
// returns an Euler circuit for G which starts at v
    {
    Let C’ = C = a circuit in G which starts and ends in v
    Let G’ = G
    Repeat
        {
        Let G’ be new graph s.t. E(G’) = E(G’)-E(C’)
            and V(G’) = V(G’) – {all isolated vertices once edges in E(C’) have been
            removed}
        If V(G’) ≠ ∅
            {
            Pick a vertex w at random from V(C) ∩ V(G’)
            Let C’ = EulerCircuit(ConnectedComponent(G’,w),w)
            C = C with C’ integrated into it at w
            }
        }
        until E(C) = E(G)
    return C
    }

    ConnectedComponent(graph G, vertex v)
    // returns the connected component of G which includes v
```